The focus of attention problem*

Dries Goossens[†] Sergey Polyakovskiy[‡] Frits C.R. Spieksma[§] Gerhard J. Woeginger[¶]

Abstract

We consider the problem of assigning sensors to track targets so as to minimize the expected error in the resulting estimation for target locations. The so-called *Focus of Attention* problem deals with the special case where every target is tracked by one pair of range sensors.

We provide a complete complexity and approximability analysis of the Focus Of Attention problem: We establish its strong NP-hardness, and we construct a polynomial time approximation scheme for it.

1 Introduction

Sensor networks offer exciting new possibilities for achieving sensory omnipresence: Tiny, inexpensive, low power, untethered, sensor devices measure and observe various environmental parameters, and thereby allow the real-time and fine-grained monitoring of physical spaces around us. In order to implement this vision, however, a number of algorithmic and combinatorial problems has to be solved. Isler, Khanna, Spletzer & Taylor [4] started the investigation of an important subarea by modelling and discussing several target tracking problems with distributed sensors. The main trouble is that (i) the used sensors are inherently limited, and individually incapable of estimating the state of a target, and that (ii) the measurements provided by these sensors are strongly corrupted by noise. Because of (i), in general a minimum of two bearing sensors is required to estimate the position of a target. Because of (ii), the choice of which measurements to combine may greatly influcence the accuracy of the system.

Among several other problems, Isler & al [4] discuss the situation where the sensors are $2n$ cameras that are to be assigned in disjoint pairs to n targets. The cameras are located on a straight line, whereas the targets are somewhere in the plane. Without loss of generality the straight line is the x-axis, so that the $2n$ cameras are positioned in the $2n$ points with coordinates $(x_i, 0)$ with $1 \le i \le 2n$. Isler & al [4] discuss an error measure motivated by stereo reconstruction that mainly depends on the y-coordinates $y_1, \dots, y_n$ of the n targets: If the ith and the jth camera together are assigned to the kth target, then the corresponding incurred error cost is

$$c_{ijk} = \frac{y_k}{|x_i - x_j|}. \tag{1.1}$$

Isler & al [4] argue that the measure in (1.1) gives a good error approximation in case the targets are not too close to the cameras. For more information on this measure and for some mathematical justifications, we refer the reader to Appendix A of [4]. The objective in the *Focus of Attention* problem is to find an assignment of (disjoint) camera pairs to targets such that the sum of all error costs c_{ijk} is minimized. We denote this optimization problem as IKST-FOA, for short.

Isler & al [4] derive a polynomial time 2-approximation algorithm for IKST-FOA. In the *equi-distant* special case of IKST-FOA, the cameras are at unit distances from each other in the $2n$ points $(i, 0)$ with $1 \le i \le 2n$. For this equi-distant special case of IKST-FOA, [4] design a very nice PTAS.

Formulation of problem p-FOA. We will investigate a certain version of the three-dimensional assignment problem that contains problem IKST-FOA as a special case. This version is based on a real parameter p, and will throughout be denoted as p-FOA. An instance of p-FOA consists of $3n$ positive real numbers $a_1, \dots, a_n$, $b_1, \dots, b_n$, and $c_1, \dots, c_n$. The cost-coefficient for a triple (i, j, k) with $1 \le i, j, k \le n$ is defined as

$$c_{ijk} = \frac{a_k}{(b_i + c_j)^p}. \tag{1.2}$$

The goal in p-FOA is to group the $3n$ numbers into n triples (where each triple contains one a_i, one b_j and one c_k) such that the sum of the cost-coefficients of these triples becomes minimum. In Section 2 we show that for $p = 1$ this problem p-FOA coincides with the classic problem IKST-FOA as discussed above.

*This research has been supported by the Netherlands Organisation for Scientific Research (NWO) under grant 639.033.403, by BSIK grant 03018 (BRICKS: Basic Research in Informatics for Creating the Knowledge Society), and by OT Grant OT/07/015.

[†]Operations Research Group, Katholieke Universiteit Leuven, Belgium. dries.goossens@econ.kuleuven.be.

[‡]Operations Research Group, Katholieke Universiteit Leuven, Belgium. sergey.polyakovskiy@econ.kuleuven.be.

[§]Operations Research Group, Katholieke Universiteit Leuven, Belgium. frits.spieksma@econ.kuleuven.be.

[¶]Department of Mathematics, TU Eindhoven, The Netherlands. gwoegi@win.tue.nl.

Our results. We completely settle the complexity and approximability of problem p-FOA. Sections 3 and 4 provide the following complexity classification of problem p-FOA:

- For every real p with $-1 \le p \le 0$, problem p-FOA is polynomially solvable.
- For every real p with $p < -1$ or $p > 0$, problem p-FOA is strongly NP-hard.
- Even the equi-distance special case of IKST-FOA is strongly NP-hard. This settles a question left open in [4].

As our main contribution, Section 5 resolves the approximability status of p-FOA:

- For every real p, problem p-FOA possesses a PTAS.

Our PTAS extends the results and ideas of [4] for the equi-distant special case. The design of our PTAS is quite intricate, and introduces a number of new ideas to the area.

Some related results. The literature contains a number of results on target tracking where cameras are to be assigned in pairs to targets. There are various ways of modelling the measurement errors and the resulting error costs. We only mention two results that have a strong algorithmic component. Gfeller, Mihalak, Suri, Vicari & Widmayer [2] discuss scenarios where the error mainly depends on the intersection angle of the two viewing cones subtended by a pair of cameras. Al-Hasan, Ramachandran & Mitchell [1] consider a related scenario with moving cameras. They introduce an intricate cost model for the movements of the cameras, and they develop a GRASP routine for cost minimization in this model.

2 IKST-FOA coincides with 1-FOA

Consider an instance of IKST-FOA that is specified by $2n$ real numbers $x_1, \ldots, x_{2n}$ and by n real numbers $y_1, \ldots, y_n$, with costs defined as in (1.1). Assume that the cameras on the x-axis are ordered as

$$x_1 \le x_2 \le \cdots \le x_{2n}.$$

A feasible solution is called *left-right separating*, if it matches every camera from the left half $1, \ldots, n$ with one camera from the right half $n+1, \ldots, 2n$ (and with some target). We stress that the essence of the following Lemma 2.1 is due to Isler & al [4].

LEMMA 2.1. *There exists an optimal solution for IKST-FOA that is left-right separating.*

Proof. A feasible solution is specified by a permutation π of $1, \ldots, 2n$, such that cameras $\pi(2k-1)$ and $\pi(2k)$ are assigned to target k for $1 \le k \le n$. Among all optimal solutions, consider one solution π that maximizes the auxiliary function $\sum_{k=1}^{n} |\pi(2k-1) - \pi(2k)|$. If π is not left-right separating, it must at least once match two cameras from the left half (say $\pi(1)$ and $\pi(2)$), and it must at least once match two cameras from the right half (say $\pi(3)$ and $\pi(4)$). Assume without loss of generality that

$$x_{\pi(1)} \le x_{\pi(2)} \le x_{\pi(3)} \le x_{\pi(4)}.$$

Then $|x_{\pi(2)} - x_{\pi(1)}| \le |x_{\pi(3)} - x_{\pi(1)}|$ and $|x_{\pi(4)} - x_{\pi(3)}| \le |x_{\pi(4)} - x_{\pi(2)}|$. Therefore switching the values $\pi(2)$ and $\pi(3)$ in π will not worsen the objective value, whereas it does increase the auxiliary function. This contradiction completes the argument.

Next, let x^* with $x_n \le x^* \le x_{n+1}$ be a real number that separates the cameras in the left half from the cameras in the right half. Then the IKST-FOA instance can be rewritten into an equivalent instance of 1-FOA in the following way: Let $a_1, \ldots, a_n$ denote the positive real numbers $y_1, \ldots, y_n$; let $b_1, \ldots, b_n$ denote the positive real numbers $x^* - x_1, \ldots, x^* - x_n$; let $c_1, \ldots, c_n$ denote the positive real numbers $x_{n+1} - x^*, \ldots, x_{2n} - x^*$. Define the cost-coefficient for a triple (i, j, k) as in (1.2).

Vice versa, every instance of 1-FOA can be rewritten into an equivalent instance of IKST-FOA, if $a_1, \ldots, a_n$ play the role of $y_1, \ldots, y_n$, and if $b_1, \ldots, b_n$ and $-c_1, \ldots, -c_n$ play the role of $x_1, \ldots, x_{2n}$. Hence IKST-FOA and 1-FOA are equivalent.

3 An NP-hardness result for p-FOA

We first recall the following formulation (3.3) of the Hölder inequality; see for instance Theorem 13 in the book [3] of Hardy, Littlewood & Pólya. For a non-zero real number q with $q < 1$, and for $2n$ positive real numbers $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ we have

$$\sum_{i=1}^{n} \alpha_i^{1/q} \, \beta_i^{(q-1)/q} \ \ge \tag{3.3} \left(\sum_{i=1}^{n} \alpha_i \right)^{1/q} \left(\sum_{i=1}^{n} \beta_i \right)^{(q-1)/q}$$

Most importantly, equality holds in (3.3) if and only if the sequences α_i and β_i are proportional, that is, if and only if there exists a real number λ such that $\alpha_i = \lambda \beta_i$ for $1 \le i \le n$. We will use these facts in the following NP-hardness proof.

THEOREM 3.1. *For all real $p < -1$ and for all real $p > 0$, problem p-FOA is strongly NP-hard.*

Proof. We reduce from the strongly NP-hard problem Numerical Matching with Target Sums (NMTS): Given target sums A_k $(1 \le k \le n)$, and given positive integers B_i $(1 \le i \le n)$ and C_j $(1 \le j \le n)$, can we find a collection of n triples (i, j, k) such that $A_k = B_i + C_j$ holds for each triple and such that each element is used exactly once? Without loss of generality, we assume that the sum $S := \sum A_k$ equals $\sum B_i + \sum C_j$.

We consider an instance of NMTS, and transform it into an instance of p-FOA by setting $a_k := A_k^{p+1}$, and $b_i := B_i$, and $c_j := C_j$ for all $1 \le i, j, k \le n$. We claim that this instance of p-FOA has a feasible solution of cost at most S, if and only if the NMTS instance has answer YES.

The if-part is easy to see: If we interpret the triples in the feasible solution for NMTS as a feasible solution for p-FOA, then any triple (i, j, k) with $A_k = B_i + C_j$ in this feasible solution contributes $A_k^{p+1}/(B_i + C_j)^p = A_k$ to the objective value. Hence, the corresponding objective value for p-FOA equals S.

For the only-if-part, we interpret the triples in the feasible solution for p-FOA with cost at most S as a feasible solution for NMTS. We use inequality (3.3) with $q = 1/(p+1)$; note that for $p < -1$ and for $p > 0$, the corresponding q indeed satisfies $q < 1$. Furthermore, we set $\alpha_i = A_k$ and $\beta_i = B_i + C_j$ in (3.3), where j and k are the indices that occur with index i in a triple (i, j, k) in the feasible solution. For the objective value this then yields

$$\begin{aligned} S &\ge \sum A_k^{p+1} \, (B_i + C_j)^{-p} \\ &\ge \left(\sum_{i=1}^{n} A_k \right)^{p+1} \left(\sum_{i=1}^{n} B_i + C_j \right)^{-p} \\ &= S. \end{aligned}$$

Therefore we are dealing with the case of equality in (3.3), and the values $\alpha_i = A_k$ and $\beta_i = B_i + C_j$ must be proportional to each other. Since $\sum \alpha_i = \sum \beta_i$, the factor λ of proportionality is $\lambda = 1$. This yields $A_k = B_i + C_j$ for all triples (i, j, k) in the feasible solution, and hence the NMTS instance has answer YES.

Finally, let us discuss the equi-distant special case of IKST-FOA where the cameras are at unit distances from each other in the $2n$ points $(i, 0)$ with $1 \le i \le 2n$. The equivalent 3AP instance with cost coefficients of the form (1.2) has $b_i = c_i = i - \frac{1}{2}$ for $i = 1, \ldots, n$.

THEOREM 3.2. *The equi-distant special case of IKST-FOA is strongly NP-hard.*

Proof. We use a similar reduction and the same notation as in the preceding theorem. Yu, Hoogeveen & Lenstra [5] have shown that Numerical Matching with Target Sums is NP-hard even if $B_i = C_i = i$ holds for $i = 1, \ldots, n$. We start with an NMTS instance A_k, B_i, C_j $(1 \le i, j, k \le n)$ of this particular form, and define a new (equivalent) NMTS instance $A'_k = A_k - 1$, $B'_i = B_i - \frac{1}{2}$ and $C'_j = C_j - \frac{1}{2}$. Then $B'_i = C'_i = i - \frac{1}{2}$, and the reduction in the preceding theorem for $p = 1$ yields the desired NP-hardness argument.

4 A polynomial time result for p-FOA

In this section we discuss the parameter range $-1 \le p \le 0$ for p-FOA with cost coefficients of the form (1.2). These problems are almost trivial. The following lemma settles the case with input sequences of length $n = 2$.

LEMMA 4.1. *Let $a_1 \le a_2$ and $b_1 \ge b_2$, $c_1 \ge c_2$ be six positive real numbers that form an instance of p-FOA with $-1 \le p \le 0$. Then the matching a_1, b_1, c_1 and a_2, b_2, c_2 forms an optimal solution.*

Proof. For any real $s \ge 0$, the function $f(x) = a_1 x^{-p} + a_2 (s - x)^{-p}$ is concave on the range $0 \le x \le s$. This implies that in the p-FOA instance the minimum cost is attained on the boundary of the domain, and that the optimal solution either matches a_1, b_1, c_1 and a_2, b_2, c_2, or a_1, b_2, c_2 and a_2, b_1, c_1. Since

$$(a_2 - a_1) \left(\frac{1}{(b_1 + c_1)^p} - \frac{1}{(b_2 + c_2)^p} \right) \ \ge \ 0,$$

the first one of these two candidate solutions gives the minimum cost.

Repeated application of Lemma 4.1 now yields the following theorem.

THEOREM 4.1. *Let $a_1 \le \cdots \le a_n$, and $b_1 \ge \cdots \ge b_n$, $c_1 \ge \cdots \ge c_n$ form an instance of p-FOA with $-1 \le p \le 0$, where the cost coefficients are of the form (1.2). Then an optimal solution is given by the triples (i, i, i) with $1 \le i \le n$.*

5 The approximability of p-FOA

In this section we discuss the approximability of p-FOA. In particular, we will derive a polynomial time approximation scheme. We will throughout concentrate on the cases with $p > 0$. In Section 5.5 we briefly sketch how to settle the remaining cases with negative p by a similar approach.

Consider an arbitrary instance I of p-FOA with cost coefficients of the form (1.2). Without loss of generality we assume that the numbers $a_1, \ldots, a_n$ are in non-decreasing order

$$(5.4) \qquad a_1 \;\le\; a_2 \;\le\; \cdots \;\le\; a_n.$$

The well-known *rearrangement inequality* (see for instance Theorem 368 in the book [3] of Hardy, Littlewood & Pólya) states the following: If two sequences $\langle \alpha_i \rangle$ and $\langle \beta_i \rangle$ are given except in arrangement, then the sum $\sum_i \alpha_i \beta_i$ is least if the two sequences are monotonic in opposite order. An immediate consequence of the rearrangement inequality and of (5.4) is that

$$(5.5) \qquad b_{\pi(1)} + c_{\sigma(1)} \;\le\; \cdots\cdots \;\le\; b_{\pi(n)} + c_{\sigma(n)}$$

holds in any reasonable feasible solution of p-FOA that consists of the triples $(\pi(k), \sigma(k), k)$ for $1 \le k \le n$. If one of the inequalities in (5.5) would be violated, then rearranging the sums $b_{\pi(k)} + c_{\sigma(k)}$ into non-decreasing order would improve the objective value. In particular, any optimal solution will satisfy (5.5).

5.1 A simple approximation algorithm for p-FOA Isler & al [4] analyze the following simple approximation algorithm for 1-FOA: For $k = 1, \ldots, n$ match the k-th largest number among $b_1, \ldots, b_n$ with the k-th smallest number among $c_1, \ldots, c_n$. Then match the resulting sums $b_i + c_j$ according to the rearrangement inequality with the numbers $a_1, \ldots, a_n$.

Isler & al [4] show that this approximation algorithm has a worst case performance guarantee of 2 for 1-FOA. This algorithm can also be applied to instances of the general p-FOA problem. By slightly modifying the arguments in [4] in a straightforward way, we get the following result.

LEMMA 5.1. *For every $p \ge 0$, the above polynomial time approximation algorithm for p-FOA has a worst case performance guarantee of 2^p.*

This bound 2^p is tight, as can be seen from the instance $a_1 = b_1 = c_1 = 1$, $a_2 = t^2$, and $b_2 = c_2 = t$ with some huge number t. Then the approximation algorithm matches the numbers $(1, 1, t)$ and $(t^2, t, 1)$, whereas the optimal solution matches the numbers $(1, 1, 1)$ and (t^2, t, t). As t tends to infinity, the ratio tends to 2^p.

For parameter values $p < -1$, the corresponding approximation algorithm would match big sums $b_i + c_j$ with small values a_k. However, the instance $a_1 = b_1 = c_1 = 1$ and $a_2 = b_2 = c_2 = t$ with huge t demonstrates that this algorithm does not have a finite performance guarantee.

5.2 Setting up the PTAS The worst case guarantee in our PTAS will be of the form $(1+\varepsilon)^{2p}$, where ε with $0 < \varepsilon < 1/2$ is a fixed real number that can be chosen arbitrarily close to zero. We introduce L as the smallest integer satisfying

$$(5.6) \qquad \varepsilon\,(1+\varepsilon)^{L-1} \;\ge\; 1.$$

Some straightforward calculations show that L is of order $O((1/\varepsilon) \ln 1/\varepsilon)$. Since ε is a constant whose value does not depend on the input, all expressions that only depend on ε and L will also be fixed constants that are independent of the size of the input.

We start with a rounding phase, in which we round down all the numbers $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ in instance I to the next integer power of $1+\varepsilon$. This rounding is harmless, since it changes the objective value by at most a factor of $(1+\varepsilon)^p$. Define K as the largest integer for which $(1+\varepsilon)^K$ occurs among these rounded values $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$. We stress that the value of K is polynomially bounded in the input size and in the reciprocal value of ε: If z is the maximum value among the $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$, then K is $O(\ln(z)/\varepsilon)$.

5.3 Definition of the auxiliary instances We introduce a family of auxiliary instances I' that encode certain useful sub-instances of the original instance I. This family has two crucial properties. First, the family is small: It contains only a polynomial number of auxiliary instances. Secondly, the auxiliary instances in this family are easy to approximate: Every instance in the family can be approximated by reducing it to several smaller instances in the family. The appropriate choice of these auxiliary instances is rather delicate, and constitutes the main step in deriving the PTAS.

Part of the structure of an auxiliary instance I' is determined by a four-tuple (m, k, β, γ) which is called the *type* of instance I'. The four-tuple consists of:

- An integer m with $1 \le m \le n$.
- An integer k with $0 \le k \le K$.
- Two non-negative integers β and γ such that $0 \le \beta, \gamma \le m$.

In the following, a real number x will be called *k-small* if $x < \varepsilon(1+\varepsilon)^{k-1}$, and it will be called *k-medium* if $\varepsilon(1+\varepsilon)^{k-1} \le x \le (1+\varepsilon)^k$. Every auxiliary instance I' of type (m, k, β, γ) consists of $3m$ real numbers $a'_1, \ldots, a'_m$, $b'_1, \ldots, b'_m$, and $c'_1, \ldots, c'_m$ that satisfy the following:

- The numbers $a'_1 \le \cdots \le a'_m$ coincide with $a_1, \ldots, a_m$, that is, they form the m smallest elements in the enumeration (5.4).

- The list $b'_1 \le \ldots \le b'_m$ consists of the β largest k-small elements among $b_1, \ldots, b_n$, together with $m - \beta$ arbitrarily chosen k-medium elements from $b_1, \ldots, b_n$.
- The list $c'_1 \le \ldots \le c'_m$ consists of the γ largest k-small elements among $c_1, \ldots, c_n$, together with $m - \gamma$ arbitrarily chosen k-medium elements from $c_1, \ldots, c_n$.
- At least one of b'_m and c'_m equals $(1+\varepsilon)^k$.

We note that for some of the types there is no corresponding auxiliary instance, as sequence $b_1, \ldots, b_n$ or sequence $c_1, \ldots, c_n$ do not contain sufficiently many k-small and k-medium elements. We also stress that the original instance I occurs among the auxiliary instances.

Let us estimate the overall number of auxiliary instances: There are $O(n^3 K)$ quadruples that describe a type. For every type (m, k, β, γ) all values a'_i, all the k-small values b'_i, and all the k-small values c'_i in any instance of that type are fixed. The k-medium values b'_i are integer powers of $1+\varepsilon$ that lie between the bounds $\varepsilon(1+\varepsilon)^{k-1}$ and $(1+\varepsilon)^k$. Inequality (5.6) yields that they must occur among the $L+1$ numbers

$$(1+\varepsilon)^{k-L},\ (1+\varepsilon)^{k-L+1},\ \cdots\cdots, (1+\varepsilon)^k.$$

Hence there are only $O(n^L)$ possible choices for the k-medium values b'_i. An analogous argument shows that there are only $O(n^L)$ possible choices for the k-medium values c'_i. Altogether this yields a polynomial upper bound of $O(K \cdot n^{2L+3})$ on the number of auxiliary instances.

5.4 Approximation of the auxiliary instances

Throughout we denote by $\mathrm{OPT}(I)$ the optimal objective value of instance I. For every auxiliary instance I', we will compute in polynomial time an approximate objective value $f(I')$ that satisfies

$$\mathrm{OPT}(I') \ \le\ f(I') \ \le\ (1+\varepsilon)^p \cdot \mathrm{OPT}(I').$$

The computation is done in order of increasing values of m: Whenever we are handling an auxiliary instance with $3m$ numbers, all auxiliary instances with $3(m-1)$ numbers have already been settled. The computation of $f(I')$ in the cases with $m=1$ is trivial.

Now consider an auxiliary instance I' of type (m, k, β, γ) with $m \ge 2$. An optimal solution matches element a'_m with two partners b^* and c^*, and the rearrangement inequality and (5.5) tell us that the sum $b^* + c^*$ of these two partners must be relatively large. Since (by the definition of an auxiliary instance) at least one of b'_m and c'_m takes the value $(1+\varepsilon)^k$, we certainly have $b^* + c^* \ge (1+\varepsilon)^k$, and this means that at least one of b^* and c^* is a k-medium element. Our strategy is to enumerate many cases, and to try out all possibilities for such a k-medium partner b^* or c^*. The case checking covers two possible scenarios.

In the first scenario both partners b^* and c^* are k-medium. Hence we check all $O(L^2)$ possibilities for b^* and c^*. In every check, we remove the corresponding three numbers a'_m, b^*, c^* from the instance I' and thus create a residual instance I'' of type $(m-1, k', \beta', \gamma')$ for appropriate integers k', β', γ'. Then $f(I'') + a'_m/(b^* + c^*)^p$ yields a $(1+\varepsilon)^p$-approximation for the objective value of the best solution that matches a'_m with b^* and c^*.

In the second scenario one partner, say the partner b^*, is k-small. Then $b^* + c^* \ge (1+\varepsilon)^k$ and $b^* < \varepsilon(1+\varepsilon)^{k-1}$ together imply $c^* > (1+\varepsilon)^{k-1}$. We conclude $b^* < \varepsilon\, c^*$, and hence

$$b^* + c^* \ \le\ (1+\varepsilon)\, c^* \ \le\ (1+\varepsilon)\,(b'_1 + c^*),$$

where b'_1 is the minimum of $b'_1, \ldots, b'_m$. Rewriting this last inequality yields

$$\frac{a'_m}{(b'_1 + c^*)^p} \ \le\ (1+\varepsilon)^p \, \frac{a'_m}{(b^* + c^*)^p}. \tag{5.7}$$

Let the instance I'' result by removing a'_m, b'_1, c^* from instance I', and let the instance I''' result by removing a'_m, b^*, c^* from instance I'. From $b'_1 \le b^*$ we derive $\mathrm{OPT}(I'') \le \mathrm{OPT}(I''')$. This yields

$$\begin{aligned} f(I'') &\le (1+\varepsilon)^p \cdot \mathrm{OPT}(I'') \\ &\le (1+\varepsilon)^p \cdot \mathrm{OPT}(I'''). \end{aligned} \tag{5.8}$$

Now how do we proceed in this second scenario? We check all $O(L)$ possibilities for a k-medium partner $c^* > (1+\varepsilon)^{k-1}$. In every single check, we match element a'_m with the elements c^* and with b'_1. The residual instance I'' then is of type $(m-1, k', \beta', \gamma')$ for appropriate integers k', β', γ'. The inequalities (5.7) and (5.8) show that

$$\begin{aligned} & f(I'') + \frac{a'_m}{(b'_1 + c^*)^p} \\ &\quad \le \quad (1+\varepsilon)^p \cdot \left(\mathrm{OPT}(I''') + \frac{a'_m}{(b^* + c^*)^p} \right) \\ &\quad = \quad (1+\varepsilon)^p \cdot \mathrm{OPT}(I'). \end{aligned}$$

Therefore the value $f(I'') + a'_m/(b'_1 + c^*)^p$ yields a $(1+\varepsilon)^p$-approximation for the objective value of the best solution that matches a'_m with b^* and c^*.

In the end, the value $f(I')$ is defined as the best approximation detected in all the explored cases under both scenarios.

5.5 The approximation scheme Let us now summarize the main steps of the approach outlined above. Consider an arbitrary instance I of p-FOA with $p > 0$.

1. Round down all $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ to the next integer power of $1 + \varepsilon$.
2. Enumerate all possible auxiliary instances I' of all possible types (m, k, β, γ).
3. Determine the value $f(I')$ for every auxiliary instance I'.
4. Output $f(I')$ for the auxiliary instance I' that coincides with the original instance I.

The running time of this approach is polynomial: The overall number of auxiliary instances is polynomially bounded by $O(K \cdot n^{2L+3})$, and every single value $f(I')$ can be computed in polynomial time. Also the approximation guarantee $(1+\varepsilon)^{2p}$ is easy to see: The rounding in Step #1 introduces a multiplicative error of at most $(1 + \varepsilon)^p$, and the computation of the function values $f(I')$ introduces another factor of at most $(1 + \varepsilon)^p$.

This yields a PTAS for p-FOA with $p > 0$. The cases with negative values of p can be settled in a very similar fashion. We modify the above PTAS in the following way: First, we reverse all inequality-signs in (5.4). Secondly, in the rounding phase instead of rounding *down* we round all the numbers *up* to the next integer power of $1 + \varepsilon$. Thirdly, in the definition of the auxiliary instances we perform two changes: For the list $b'_1 \leq \ldots \leq b'_m$ we now choose the β smallest (and not the β largest) k-small elements among $b_1, \ldots, b_n$, and for the list $c'_1 \leq \ldots \leq c'_m$ we choose the γ smallest k-small elements among $c_1, \ldots, c_n$. Finally, in the second scenario in Section 5.4 we do not match element a'_m with the elements c^* and the smallest k-small element b'_1, but we match a'_m with c^* and with the largest k-small element. The rest of the analysis goes through just as before, and we leave all details to the reader.

THEOREM 5.1. *For all real p, problem p-FOA possesses a PTAS.*

References

[1] M. AL-HASAN, K.K. RAMACHANDRAN, AND J.E. MITCHELL (2008). Optimal placement of stereo sensors. *Optimization Letters 2*, 99–111.

[2] B. GFELLER, M. MIHALAK, S. SURI, E. VICARI, AND P. WIDMAYER (2008). Angle optimization in target tracking. *Proceedings of the 11th Scandinavian Workshop on Algorithm Theory (SWAT'2008)*, Springer LNCS 5124, 65–76.

[3] G.H. HARDY, J.E. LITTLEWOOD, AND G. PÓLYA (1934). *Inequalities.* Cambridge University Press.

[4] V. ISLER, S. KHANNA, J.R. SPLETZER, AND C.J. TAYLOR (2005). Target tracking with distributed sensors: The focus of attention problem. *Computer Vision and Image Understanding 100*, 225–247.

[5] W. YU, H. HOOGEVEEN, AND J.K. LENSTRA (2004). Minimizing makespan in a two-machine flow shop with delays and unit-time operations is NP-hard. *Journal of Scheduling 7*, 333–348.